Project systems theory

Final exam 2017–2018, Thursday 25 January 2018, 9:00 – 12:00

Problem 1 $(3 + 3 + 8 = 14$ points)

A simple model of a magnetic levitation system is given as

$$m\ddot{q}(t) = mg - \frac{1}{2}\frac{L}{(1+q(t))^2}u^2(t), \tag{1}$$

with $q(t)$ the position of the levitated mass with mass $m > 0$ and $g > 0$ the gravitational constant. The input current to the electromagnet that suspends the mass is denoted by $u(t)$ and $L > 0$ is a constant.

(a) Write the system (1) in the form of a nonlinear state-space system $\dot{x} = f(x, u)$ by taking $x_1(t) = q(t)$ and $x_2(t) = \dot{q}(t)$.

(b) Let $\bar{x} = [\,\bar{q}\ 0\,]^{\mathrm{T}}$ be the desired equilibrium point for some $\bar{q} > 0$. Give the constant input $u(t) = \bar{u}$ with $\bar{u} > 0$ that achieves this equilibrium point.

(c) Linearize the state-space system around the equilibrium point given by $\bar{x}$ and $\bar{u}$.

Problem 2 (14 points)

Consider the polynomial

$$p(\lambda) = \lambda^4 + 3\lambda^3 + \lambda^2 + a\lambda + 2a, \tag{2}$$

where $a \in \mathbb{R}$. Determine the values of a for which the polynomial is stable.

Problem 3 $(4 + 12 + 6 = 22$ points)

Consider the system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \tag{3}$$

with state $x(t) \in \mathbb{R}^2$, output $y(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -6 & 3 \\ -7 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 1 \end{bmatrix}. \tag{4}$$

(a) Is the system observable?

(b) Find a nonsingular matrix T and real numbers a_1, a_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{5}$$

Hint. Recall that the pair (A, C) is observable if and only if the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is controllable.

(c) Find a matrix G such that the matrix $A - GC$ has eigenvalues -2 and -4.

Note. Such matrix G characterizes a stable state observer

$$\dot{w}(t) = Aw(t) + G(y(t) - Cw(t)) = (A - GC)w(t) + Gy(t). \tag{6}$$

Problem 4 (4 + 4 + 7 + 7 = 22 points)

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 8 & 2 & -7 \\ -12 & -2 & 14 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t). \tag{7}$$

(a) Is the system (7) (asymptotically) stable?

(b) Determine the reachable subspace of the system (7) and give a basis for this subspace.

For the remainder of this problem, consider the system

$$\dot{x}(t) = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & b & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ b \end{bmatrix} u(t), \tag{8}$$

where a and b are real parameters.

(c) Determine all values a, b for which the system (8) is controllable.

(d) Determine all values a, b for which the system (8) is stabilizable.

Problem 5 (4 + 14 = 18 points)

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{9}$$

and recall the general solution for a given initial condition $x_0 \in \mathbb{R}^n$ and input $u(\cdot)$ as

$$x_u(t, x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s. \tag{10}$$

Assume that the system (9) is controllable and define the input signal

$$\bar{u}(t) = B^{\mathrm{T}}e^{-A^{\mathrm{T}}t}K^{-1}e^{-AT}x_T \tag{11}$$

for some $x_T \in \mathbb{R}^n$ and fixed $T > 0$ and where

$$K = \int_0^T e^{-As}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}s}\,\mathrm{d}s. \tag{12}$$

(a) Let $x_0 = 0$. Show that the application of the input $\bar{u}(\cdot)$ leads to $x_{\bar{u}}(T, 0) = x_T$. Here, you can assume that K^{-1} exists.

(b) Show that K is nonsingular.

(10 points free)